Propagation of Instrumental Errors to the Sky Temperature Measurement

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This report studies the impact on the sky temperature $T_{sky}$, of deviations in different instrumental parameters from their nominal values.

An idealized instrumental model is assumed, described by equation 13 of the Rogers & Bowman (2012)\(^1\) paper.

The analysis is performed in the range 100-199 MHz.

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Equation for the Sky Temperature

From Equation 13 of the paper:

\[
T_{sky} = \left[ \frac{1 - |\Gamma_l|^2}{(1 - |\Gamma_a|^2)|F|^2} \right] \times
\left[ T_{amb} + T_{cal} \frac{(P_{ant} - P_{load})}{(P_{cal} - P_{load})} - T_u \frac{|\Gamma_a|^2|F|^2}{(1 - |\Gamma_l|^2)} \right.
- \left. (T_c \cos \phi_a F + T_s \sin \phi_a F) \frac{|\Gamma_a||F|}{(1 - |\Gamma_l|^2)} \right]
\]  \tag{1}

\[\begin{align*}
\text{\(\uparrow\quad F\)} & = \sqrt{\frac{1 - |\Gamma_l|^2}{1 - \Gamma_a \Gamma_l}} \\
\text{\(\uparrow\quad \phi_a F\)} & = \text{phase}(\Gamma_a \cdot F)
\end{align*}\]
Methods

Errors are expected on the following vector parameter:

\[ \bar{\theta} = (T_{amb}, T_{cal}, T_c, T_s, T_u, |\Gamma_a|, \phi_a, |\Gamma_l|, \phi_l) \]  (2)

These errors are propagated to the sky temperature using two methods:

1. Calculus-based approach (Taylor expansion, assuming small errors)
2. Perturbation-based approach

Both cases require the use of the nominal value and the error assigned to each of these parameters.

Computations are performed at a frequency resolution of 1 MHz, and in some cases interpolation or model-fitting are necessary to take the data to this resolution.

The following slides present the nominal values of the parameters along with a brief description.
Figure: Nominal values of 300 K and 400 K are used for the ambient and excess temperatures.
Figure: These values and slopes were obtained from the Rogers & Bowman (2012) paper, figure 8. They were projected to the 100-199 MHz range at 1 MHz resolution.
Figure: These spectra were obtained from current EDGES data. Interpolation was used to reduce the frequency resolution to 1 MHz. Some noise remains after interpolation.
Figure: The reflection of the ferrite-balun antenna is used, expected to be close to the one in the field. It is a measurement performed last year, no interpolation or smoothing involved. The LNA profile was provided by Hamdi. The measurement was very noisy, and therefore modeled using a 5th-order polynomial for smoothing purposes. The red trace is the model.
Fiducial Values: Antenna and LNA Phase

Figure: Same as previous plot.
Description of Method 1

- Vector parameter

\[ \bar{\theta} = (T_{amb}, T_{cal}, T_c, T_s, T_u, |\Gamma_a|, \phi_a, |\Gamma_l|, \phi_l) \]

- Sky temperature

\[ T_{sky} = f(\bar{\theta}) \]  \hspace{1cm} (3)

- Error

\[ \sigma^2_{sky} = \sum_{i=1}^{9} \left( \frac{\partial f}{\partial \theta_i} \right)^2 \sigma^2_{\theta_i} + \sum_{i=1}^{9} \sum_{k \neq i} 2 \frac{\partial f}{\partial \theta_i} \frac{\partial f}{\partial \theta_k} \sigma_{\theta_i \theta_k} \]  \hspace{1cm} (4)
Description of Method 1: Matrix Form

\[
\sigma_{\text{sky}}^2 = \left( \frac{\partial f}{\partial \theta_1} \quad \frac{\partial f}{\partial \theta_2} \quad \ldots \quad \frac{\partial f}{\partial \theta_9} \right) \begin{pmatrix} \sigma_{\theta_1}^2 & \sigma_{\theta_1 \theta_2} & \ldots & \ldots \\ \sigma_{\theta_2 \theta_1} & \sigma_{\theta_2}^2 & \ldots & \ldots \\ \ldots & \ldots & \sigma_{\theta_9}^2 & \ldots \\ \ldots & \ldots & \ldots & \sigma_{\theta_9}^2 \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial \theta_1} \\ \frac{\partial f}{\partial \theta_2} \\ \ldots \\ \frac{\partial f}{\partial \theta_9} \end{pmatrix}
\]

\[
\sigma_{\text{sky}}^2 = \mathbf{J} \cdot \mathbf{C} \cdot \mathbf{J}^T \tag{5}
\]

- The Jacobian matrix $\mathbf{J}$ is computed symbolically, and then evaluated at the fiducial vector parameter $\bar{\theta}_0$, at each frequency.
- The covariance matrix $\mathbf{C}$ is generated assuming realistic errors.
- Errors are assigned for one parameter at a time, in order to see their individual impact on the sky temperature.
- No covariance is assigned.
Description of Method 2

In the second method, the propagation of errors is performed as:

\[ \Delta T_{\text{sky}} = f(\bar{\theta}^*) - f(\bar{\theta}_0) \]  

\( f(\bar{\theta}) = T_{\text{sky}} \)

\( \bar{\theta}_0 \): fiducial vector parameter

\( \bar{\theta}^* \): vector parameter after perturbation of one parameter
Comparison of Methods

The following plots show a comparison of the two methods.

In Method 1, the error in parameter $\theta_i$ is characterized by a standard deviation $\sigma_i$. The output error is also a standard deviation, always equal or larger than zero.

In Method 2, the input and output are differences with respect to the fiducial values. Since the output difference can be positive or negative, the plots show its absolute value. This simplifies the comparison between the methods.

The assigned errors (or perturbations) are constant in frequency. Two values have been chosen for each parameter, as follows:

- Temperatures: 1 K and 5 K
- Magnitude of reflection coefficient: 0.000115 and 0.00115 (0.01 dB and 0.1 dB at -20 dB)
- Phase of reflection coefficient: 0.1° and 1.0°.
\[ \Delta T_{\text{ambient}} = 1 \text{ [K]} \]

**Figure:** Results with the two methods overlap.
Figure: Results with the two methods overlap.
$\Delta T_{\text{excess}} = 1 \text{ [K]}$

**Figure:** Results with the two methods overlap. Noise is propagated from that of the spectra (page 7).
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Figure: Results with the two methods overlap.
\[ \Delta T_{\text{cosine}} = 5 \text{ [K]} \]

Figure: Results with the two methods overlap.
$\Delta T_{\text{sine}} = 1 \text{ [K]}$

**Figure:** Results with the two methods overlap.
$\Delta T_{\text{sine}} = 5 \text{ [K]}$

**Figure**: Results with the two methods overlap.
\[ \Delta T_{\text{uncorr}} = 1 \, [\text{K}] \]

**Figure:** Results with the two methods overlap.
\[ \Delta T_{\text{uncorr}} = 5 \text{ [K]} \]

Figure: Results with the two methods overlap.
Figure: Differences between methods are evident for the antenna reflection coefficient. They are expected, since equation 1 is non-linear in this parameter. Shape is similar, however, which serves as validation.
\[ \Delta |\Gamma_{\text{ant}}| = 0.00115 \text{ (0.10 dB @ −20 dB)} \]

Figure: Differences between methods are evident for the antenna reflection coefficient. They are expected, since equation 1 is non-linear in this parameter. Shape is similar, however, which serves as validation.
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Summary

The following table summarizes the results, obtained with method 2. This method is preferred since its output error is allowed to be positive or negative.

The RMS value is computed in the range 100-199 MHz, in four ways: 1) direct, 2) after removing an offset, 3) after removing the best-fit function of the form \( k \cdot x^\alpha \) (\( k \) and \( \alpha \) are fit parameters), and 4) after removing the best-fit function of the form \( k_1 \cdot x^\alpha + k_2 \) (\( k_1 \), \( k_2 \), and \( \alpha \) are fit parameters).

The results due to errors in reflection coefficients are compared to those presented in Alan’s memo 105, for the case with no attenuation between the switch and the LNA (first column of his first table). His results represent the range 50-200 MHz, but this was the closest configuration I found to compare to my results. His results due to temperature errors are not directly comparable to mine, so I do not show them here.

<table>
<thead>
<tr>
<th>source error</th>
<th>magnitude</th>
<th>RMS residuals in mK in 100-199 MHz</th>
<th>Alan’s</th>
</tr>
</thead>
<tbody>
<tr>
<td>antenna ( S_{11} )</td>
<td>0.01 dB</td>
<td>direct 36 offset 29 scale 25 scale + offset 22</td>
<td>63</td>
</tr>
<tr>
<td>antenna ( S_{11} )</td>
<td>1°</td>
<td>direct 268 offset 259 scale 227 scale + offset 222</td>
<td>417</td>
</tr>
<tr>
<td>LNA ( S_{11} )</td>
<td>0.1 dB</td>
<td>direct 201 offset 187 scale 173 scale + offset 140</td>
<td>345</td>
</tr>
<tr>
<td>LNA ( S_{11} )</td>
<td>1°</td>
<td>direct 240 offset 237 scale 217 scale + offset 210</td>
<td>398</td>
</tr>
<tr>
<td>ambient</td>
<td>1 K</td>
<td>direct 1084 offset 132 scale 106 scale + offset 104</td>
<td></td>
</tr>
<tr>
<td>excess</td>
<td>1 K</td>
<td>direct 295 offset 291 scale 226 scale + offset 73</td>
<td></td>
</tr>
<tr>
<td>cosine</td>
<td>1 K</td>
<td>direct 245 offset 210 scale 202 scale + offset 198</td>
<td></td>
</tr>
<tr>
<td>sine</td>
<td>1 K</td>
<td>direct 127 offset 127 scale 127 scale + offset 127</td>
<td></td>
</tr>
<tr>
<td>uncorrelated</td>
<td>1 K</td>
<td>direct 107 offset 89 scale 82 scale + offset 76</td>
<td></td>
</tr>
</tbody>
</table>

As the table shows, my results are comparable to Alan’s (first four rows). In fact they are always smaller by a factor of 2 approximately. As expected, the errors decrease as more parameters are used to model them. For equal error in temperature, the smallest impact is obtained for the excess and uncorrelated temperatures.
Conclusion

- The impact of instrumental errors on the measured sky temperature has been studied, although for a not-so-realistic instrumental model.
- Two methods of error propagation were used, for purposes of comparison and validation. They provide very similar results for the level of errors assigned to the parameters.
- The nominal (or fiducial) values for some terms of equation 1 had to be interpolated or modeled, in order to use a uniform frequency resolution, and to appreciate the effect of a constant absolute error in the parameters on the sky temperature, with noise remaining sub-dominant.
- The results for the antenna and LNA reflection coefficients are comparable to those presented by Alan in memo 105, although smaller by a factor of 2.
- My analysis is evolving in order to determine the parameter estimation capabilities of EDGES in the presence of sky noise, instrument noise, and foregrounds.