MWA REVB LNA Measurements

Hamdi Mani, Judd Bowman

Abstract—The MWA LNA (REVB) was measured on the Low Frequency Radio astronomy Lab using state of the art test equipment. S-parameters of the amplifier were measured, gain and impedance were computed. The noise of the amplifier was also measured using an RFI tight enclosure to reject RFI. It was found that the amplifier needed additional shielding to prevent RFI pickup and interaction with other neighboring components (LNA of the other polarization).

Index Terms—radio telescope arrays, Dipole antennas, differential, common and mixed mode s-parameters, Low noise amplifiers, noise measurement, differential amplifier, VHF instrumentation.

I. INTRODUCTION

The MWA LNA is connected to the dipole antenna using a twin lead balanced transmission line. The LNA has a balanced input and an un-balanced 50 ohm output. The DC bias is injected from the output connector using a bias tee. The design is based on a single stage differential pair of the Avago ATF35143 enhancement mode low noise HEMT [1]. The transistor is a surface mount component; the board is made on FR4 material.

II. DIFFERENTIAL AMPLIFIER IMPLEMENTATIONS

There are few ways to realize differential amplifier:

1. 2 single ended matched amplifier stages followed by a 180deg, 3dB coupler (or balun)
2. 2 single ended matched amplifier stages preceded by a 180deg, 3dB coupler (or balun)
3. "Long tail pair": same circuit topology than what is commonly used in operational amplifier design: 2 matched transistors with common source.

The circuit topology used for the MWA LNA is: 2matched single ended amplifier stage combined in and out of phase using a transformer into a single ended output.

III. S-PARAMETER MEASUREMENT

The amplifier has a 50 Ohm SMA as RF output/ DC input. The input consisted of microstrip lines on FR4 board where the 2 twin leads connecting to the antenna are soldered. The twin lead connection cannot be used to connect to test equipment to do the measurement; it was also found that the impedance of the twin leads change when you change their shapes.

SMA connectors are mounted on the input of each LNA. The center pin of the SMA is soldered to the microstrip input lines and the jacket of the SMA is soldered to the GND layer (solder mask was removed). This transition can be thought of as a vertical microstrip to coax transition. Fig1 shows the amplifier with the SMA connectors.

Fig1. Top right: original PCB. Top left: solder mask removed to access the ground plane. Bottom right: center pin of SMA soldered to the input lines of the LNA. Bottom left: SMA jacket soldered to the GND metal layer of the PCB.

The LNA is 2-port network composed of a balanced input port and a single ended output port 2. To measure the S-parameters of the LNA, it can be considered as a 3 single ended port network. The balanced input port is made of ports 1 and 2 as shown on Fig2.

Measuring the s parameters of the LNA is reduced to measuring the 9 s-parameters of the 3 single ended networks.
The R&S ZVL3 VNA was used with these settings:
- Start Frequency: 5MHz
- Stop Frequency: 3000MHz
- Frequency Points: 601 point
- Test Port Power: -40dBm (for 2 Ports)
- IF Bandwidth: 500Hz (Lower IF BW at lower frequencies)
- Averaging: 10 sweeps
- Calibration: SOLT using HP 85052B Cal Kit
- Bias Tee at the output; calibrated out.

Since this is the first measurement done with this newly acquired network analyzer, the same measurements were done with the Agilent PNA-L N5230A:
- Start Frequency: 10MHz
- Stop Frequency: 1000MHz
- Frequency Points: 600 point
- Test Port Power: -40dBm
- IF Bandwidth: 500Hz (Lower IF BW at lower frequencies)
- Averaging: 10 sweeps
- Calibration: SOLT using HP 85052B Cal Kit
- Bias Tee at the output calibrated out.

The length of SMA connector is 23mm in Air (measured) and was corrected for.

The LNA was biased at 5.5V, 118mA total. The voltage measured on the output SMA 5V (0.5v voltage drop across the bias tee), the gate to source voltage for both transistors is 0.56V and the drain to source voltage is 2.98V.

The single ended S11, S22 and S33 are shown on the smith chart of Fig3. S11 and S22 are almost equal on the 80-300MHz band; the amplifier pair is well balanced.

The single ended gains of the 2 amplifier pairs are shown on Fig4 and the single ended reverse gain is shown on Fig6. The isolation (or coupling) between the 2 single ended inputs is about 20dB.
The measured single ended s-parameters need to be converted to differential, common and mixed mode s-parameters to describe the behavior of the LNA to different mode signals. Fig 7. Shows the needed s-parameters for this network.

Fig 7. Mixed mode s-parameters for a 2 port network where: Port 1 is balanced and port 2 is single ended. The network is fully characterized by these 9 parameters.

The different parameters are computed in Microwave office (MWO) [4] design environment using the “MMCONV” mixed mode converter function built into MWO.

Fig 8 shows the circuit used to compute the differential mode input reflection coefficient Sdd11 and the actual result (Fig 9).

Fig 9. Sdd11 relative to 100 Ohm computed from measured single ended s-parameters.

Fig 9 is the input return loss of the LNA relative to 100 Ohm impedance.

Fig 10. Is the return loss relative to 200 Ohm impedance.
Fig 11. Complex Sdd11 of the LNA on a Smith Chart. The markers display de-normalized impedance.

Fig 12. Real and Imaginary parts of the input differential impedance

Fig 13. Return loss of the LNA output showing good impedance match to 50 Ohm.

Fig 14. Common mode to differential conversion parameter showing low conversion factor.

Fig 15. Common mode input reflection coefficient (relative to 25 Ohm) showing reflective rejection of common mode signals at the input of the LNA.

Fig 16. Differential to common mode conversion parameter showing low conversion factor (low EMI generation)
Fig 17. Differential mode Gain $S_{dd21}$

Fig 18. $S_{cc21}$ common mode noise at balanced port 1 transferred to single ended port 2

Fig 19. Reverse Differential Gain showing good isolation.

Fig 20. Reverse common mode Gain showing good isolation.

Fig 21. Common Mode Rejection Ratio (CMRR) shown with the Differential and Common mode Gains.

The Common mode rejection ratio is the ratio of the differential to common mode gains of the amplifier [5]:

$$\text{CMRR [dB]} = 20 \times \log_{10}(\frac{|S_{dd21}|}{|S_{cc21}|})$$

or:

$$\text{CMRR [dB]} = S_{dd21} [\text{dB}] - S_{cc21} [\text{dB}]$$

The CMRR varies between 24dB and 15.2dB on the 80-300MHz band.

IV. S-PARAMETERS MEASUREMENTS ON A WIDE BANDWIDTH

To check the stability of the amplifier outside its operating band, an s-parameter measurement was made using the ZVA24 VNA. The measurement settings:

- 10MHz – 24GHz
- Port Power: -40dBm
- SOLT CAL
- 3 ports calibrated: Port 1, Port 3 and Port 2
- AVG: 100 averages.

Low signal power was used to keep the LNA operating on its linear range. The signal to noise ratio at the VNA receivers is low that is why 100 sweeps were averaged, this will reduce trace noise.
The magnitude of the input reflection coefficient of the MWA LNA REV B measured to 24GHz. Sdd11[dB] is < 0 up to 24 GHz.

The magnitude of the output reflection coefficient of the MWA LNA. Sss22[dB] is < 0 up to 24 GHz, the positive Sss22 at about 20GHz is due to a lot of trace noise probably caused by low signal strength reaching the measurement receiver of the VNA. Similar trace noise levels are seen on Sdd11 curve.

Differential Gain of the LNA.

Reverse gain of the LNA showing good output to input isolation.

Common mode gain of the amplifier shows high CM gain out of the MWA band. The CM gain peaks at 1.2GHz. The relatively high common mode gain should not be a problem since the antenna has high common mode reflection coefficient (Scc11) which means most of the common mode signals will be reflected at the antenna ports.

The Rolett’s stability factor (K-factor) was computed from the measured s-parameters. The necessary and sufficient conditional for unconditional stability (stability at all possible input and output impedances): $K > 1$ and $B1 > 0$

$$K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{21}|^2}$$

$$\Delta = S_{11}S_{22} - S_{12}S_{21}$$

$$B1 = 1 + |S_{11}|^2 + |S_{22}|^2 - |\Delta|^2$$

$\Delta$ is the determinant of the s-matrix and B1 is the auxiliary stability factor.
The input and output stability circles are shown below but all the instability circles are outside the Smith chart since the LNA is unconditionally stable.

**Fig. Circuit used to compute the differential mode stability factor.**

**Fig. Input stability circles.**

**Fig. K-factor of the MWA LNA showing unconditional stability up to 24GHz since K<1.**

**Fig. Output stability circles.**

**Fig. Auxiliary stability factor B1: B1 >0.**
The ASU Low Frequency Radio Astronomy & Experimental Cosmology Lab
Internal Report # 03
12/30/2010

The measured LNA sparameters were used together with the measured antenna impedance to compute the transducer power gain of the amplifier. The Ssd21 parameter shown on Fig 17 is the power gain of the amplifier when its input (balanced input) is terminated with 100 Ohm impedance and when its output (unbalanced) is terminated with a 50 ohm impedance. The input of LNA is connected to the dipole antenna through wire terminals or twin leads. The impedance of the antenna was measured before and data reported on the “MWA antenna impedance Memo”.

The figure below shows the differential gain of the amplifier when its input is terminated with 100 ohm and when it is terminated with the antenna impedance.

**VI. NOISE MEASUREMENTS**

The noise of the MWA LNA was measured using a 3dB 180deg coupler or balun that transforms the balanced input of the LNA to unbalanced port where a calibrated HP346C noise source was connected. The noise output of the noise generator
was too high and its output impedance changes between ON/OFF states. A 10dB pad was inserted between the noise source and the balun to improve the match. Since the LNA was a bare printed circuit board with no metal enclosure, an RFI tight enclosure was used to do the measurement to avoid RFI pick up which can saturate the amplifier and add noise to the data. A bias tee was connected to the output of the amplifier to inject bias. The setup schematics are shown on Fig22.

Fig22. setup used to do noise measurement.

A picture of the actual setup is shown on Fig23:

Fig23. Noise measurement setup.

It was found out that the bias tee allows the FM band signals to leak into the amplifier. This was caused by a low bypass capacitance on the DC port of the bias tee, a capacitor was mounted on the bias tee to cut the FM band signals. Fig24. shows an inside view of the modified bias tee.

Fig24. Mini circuits bias tee with an additional capacitor on the Dc port. The loss of the Balun, 10dB attenuator and cabling was measured using a 3 port VNA and was used to correct the measured noise.

Fig25 shows a plot of the noise source ENR (excess noise ratio) and a the Y-factor:

Fig25. Top: ENR of the noise source: there is only one calibration frequency point on the band, the ENR table was interpolated to do the measurement since it is assumed the signal output is flat. Bottom: the Yfactor = Phot/Pcold  when the noise source was turned ON/OFF. The Yfactor on this figure was corrected for the noise of a post amplifier used
between the LNA under test and the spectrum analyzer to improve the signal to noise ratio at the input of the spectrum analyzer.

The measured noise of the LNA including the cables, 10dB pad and balun is shown on Fig26.

![Noise Temperature Graph]

Fig26. Uncorrected noise temperature of the LNA: this noise includes the losses of the cables and the balun.

The measured loss of cables, pad and balun was used to get the noise of the LNA by itself.

![MWA LNA REV B Noise Measurement with a Balun]

Fig27. Plot showing the corrected noise of the LNA

VII. CONCLUSION & DISCUSSION

VIII. ACCOMPANYING FILES

AKNOWLEDGEMENT

REFERENCES


**Differential, Common, Single and Mixed Mode S-Parameters**

**OF A BALANCED INPUT / UNBALANCED OUTPUT 2-PORT NETWORK**

**SINGLE-ENDED 2-PORT NETWORK**

The differential mode voltage between the two terminals of a balanced port is defined as:

\[ V_d = V_1 - V_2 \]

The differential mode current is defined as:

\[ I_d = \frac{1}{2} (I_1 - I_2) \]

The differential mode impedance is then:

\[ Z_d = \frac{V_d}{I_d} = 2 \cdot Z_0 \]

**BALANCED/UNBALANCED OUTPUT 2-PORT NETWORK**

The common mode voltage across the two terminals is defined as:

\[ V_c = \frac{1}{2} (V_1 + V_2) \]

The common mode current is:

\[ I_c = I_1 + I_2 \]

The common mode impedance:

\[ Z_c = \frac{V_c}{I_c} = \frac{Z_0}{2} \]

\[ Z_{dc} = Z_0 \]

\[ Z_{dc} = 2 \cdot Z_0 \]

\[ a_1, a_2 \text{ and } a_3 \text{ are defined as:} \]

\[ a_1 = \frac{1}{Z_0} (V_1 + Z_0 \cdot I_1) \]

\[ a_2 = \frac{1}{Z_0} (V_2 + Z_0 \cdot I_2) \]

\[ a_3 = \frac{1}{Z_0} (V_3 - Z_0 \cdot I_3) \]

\[ b_1, b_2 \text{ and } b_3 \text{ are defined as:} \]

\[ b_1 = \frac{1}{2Z_0} (V_1 - Z_0 \cdot I_1) \]

\[ b_2 = \frac{1}{2Z_0} (V_2 - Z_0 \cdot I_2) \]

\[ b_3 = \frac{1}{2Z_0} (V_3 - Z_0 \cdot I_3) \]

**REFERENCES**

**Response**

\[
\begin{bmatrix}
    b_{d1} \\
    b_{c1} \\
    b_{d2}
\end{bmatrix}
= \begin{bmatrix}
    S_{dd1} & S_{dc1} & S_{dc1} \\
    S_{cd1} & S_{cc1} & S_{cc1} \\
    S_{cd1} & S_{cd1} & S_{cd1}
\end{bmatrix}
\begin{bmatrix}
    a_{d1} \\
    a_{c1} \\
    a_{d1} + a_{c1}
\end{bmatrix}
\]

**Excitation Waves**

\[
\begin{align*}
    b_{d1} &= S_{dd1} \cdot a_{d1} + S_{dc1} \cdot a_{c1} + S_{dc1} \cdot a_{d1} \\
    b_{c1} &= S_{cd1} \cdot a_{d1} + S_{cc1} \cdot a_{c1} + S_{cc1} \cdot a_{c1} \\
    b_{d2} &= S_{cd1} \cdot a_{d1} + S_{cd1} \cdot a_{c1} + S_{cd1} \cdot a_{d1}
\end{align*}
\]

\[
\begin{align*}
    S_{dd1} &= \frac{b_{d1}}{a_{d1}} \left| \begin{array}{c}
        a_{c1} = 0 \\
        a_{d1} = 0
    \end{array} \right| \\
    S_{dc1} &= \frac{b_{d1}}{a_{c1}} \left| \begin{array}{c}
        a_{d1} = 0 \\
        a_{d1} = 0
    \end{array} \right| \\
    S_{ds1} &= \frac{b_{d1}}{a_{d1}} \left| \begin{array}{c}
        a_{d1} = 0 \\
        a_{c1} = 0
    \end{array} \right|
\end{align*}
\]

\[
\begin{align*}
    S_{cd1} &= \frac{b_{c1}}{a_{d1}} \left| \begin{array}{c}
        a_{c1} = 0 \\
        a_{d1} = 0
    \end{array} \right| \\
    S_{cc1} &= \frac{b_{c1}}{a_{c1}} \left| \begin{array}{c}
        a_{d1} = 0 \\
        a_{d1} = 0
    \end{array} \right| \\
    S_{cs1} &= \frac{b_{c1}}{a_{d1}} \left| \begin{array}{c}
        a_{d1} = 0 \\
        a_{c1} = 0
    \end{array} \right|
\end{align*}
\]

\[
\begin{align*}
    S_{dd2} &= \frac{b_{d2}}{a_{d1}} \left| \begin{array}{c}
        a_{c1} = 0 \\
        a_{d1} = 0
    \end{array} \right| \\
    S_{dc2} &= \frac{b_{d2}}{a_{c1}} \left| \begin{array}{c}
        a_{d1} = 0 \\
        a_{d1} = 0
    \end{array} \right| \\
    S_{ds2} &= \frac{b_{d2}}{a_{d1}} \left| \begin{array}{c}
        a_{d1} = 0 \\
        a_{c1} = 0
    \end{array} \right|
\end{align*}
\]

- \( a_{d1} \) and \( a_{c1} \) are the differential and common mode respectively normalized power waves incident on balanced port 1.
- \( a_{d2} \) is the normalized power wave incident on single ended port 2.

\[
\begin{align*}
    a_{d1} &= \frac{1}{2\sqrt{Z_{0d}}} \left( V_{d1} + Z_{0d} \cdot I_{d1} \right) \\
    a_{c1} &= \frac{1}{2\sqrt{Z_{0c}}} \left( V_{c1} + Z_{0c} \cdot I_{c1} \right) \\
    a_{d2} &= \frac{1}{2\sqrt{Z_{0d}}} \left( V_{d2} + Z_{0d} \cdot I_{d2} \right)
\end{align*}
\]

\[
\begin{align*}
    b_{d1} &= \frac{1}{2\sqrt{Z_{0d}}} \left( V_{d1} - Z_{0d} \cdot I_{d1} \right) \\
    b_{c1} &= \frac{1}{2\sqrt{Z_{0c}}} \left( V_{c1} - Z_{0c} \cdot I_{c1} \right) \\
    b_{d2} &= \frac{1}{2\sqrt{Z_{0d}}} \left( V_{d2} - Z_{0d} \cdot I_{d2} \right)
\end{align*}
\]

\[
\begin{align*}
    b_{d1} &= \frac{1}{2\sqrt{Z_{0d}}} \left( V_{d1} - Z_{0d} \cdot I_{d1} \right) \\
    b_{c1} &= \frac{1}{2\sqrt{Z_{0c}}} \left( V_{c1} - Z_{0c} \cdot I_{c1} \right) \\
    b_{d2} &= \frac{1}{2\sqrt{Z_{0d}}} \left( V_{d2} - Z_{0d} \cdot I_{d2} \right)
\end{align*}
\]

- \( b_{d1} \) and \( b_{c1} \) are the differential and common mode normalized power waves reflected from balanced port 1.
- \( b_{d2} \): reflected wave from un-bal port 2.
\[\begin{align*}
\alpha_d &= \frac{1}{2 \sqrt{2} \Delta_0} \left( V_0 + Z_0 \cdot I_d \right) \\
&= \frac{1}{2 \sqrt{2} \sqrt{Z_0}} \left( V_1 - V_2 + Z_0 \cdot I_0 \right) \\
&= \frac{1}{2 \sqrt{2} \sqrt{Z_0}} \left[ V_1 + Z_0 \cdot I_1 - \left( V_2 + Z_0 \cdot I_2 \right) \right] \\
&= \frac{1}{\sqrt{2}} \left( a_1 - a_2 \right)
\end{align*}\]

\[\begin{align*}
\beta_d &= \frac{1}{2 \sqrt{2} \Delta_0} \left( V_d - Z_0 \cdot I_d \right) \\
&= \frac{1}{2 \sqrt{2} \sqrt{Z_0}} \left( V_1 - V_2 + Z_0 \cdot I_0 \right) \\
&= \frac{1}{\sqrt{2}} \left[ \frac{1}{2 \sqrt{2} \sqrt{Z_0}} \left( V_1 - Z_0 \cdot I_1 \right) - \frac{1}{2 \sqrt{2} \sqrt{Z_0}} \left( V_2 - Z_0 \cdot I_2 \right) \right] \\
&= \frac{1}{\sqrt{2}} \left( b_1 - b_2 \right)
\end{align*}\]

\[\begin{align*}
\gamma_d &= a_d + a_c \\
\beta_c &= b_d + b_c \\
\alpha_c &= \frac{2}{\sqrt{2}} \alpha_1 \\
\alpha_1 &= \frac{1}{\sqrt{2}} \left( \alpha_d + \alpha_c \right) \\
\alpha_2 &= \frac{1}{\sqrt{2}} \left( \alpha_d - \alpha_c \right)
\end{align*}\]

\[\begin{align*}
\alpha_4 &= \frac{a_3}{a_2} \\
b_4 &= b_3
\end{align*}\]

\[\begin{align*}
q_1 &= q_3 \\
b_1 &= b_3 + b_2
\end{align*}\]

\[\begin{align*}
a_1 &= \frac{1}{\sqrt{2}} \left( a_d + a_c \right) \\
b_1 &= \frac{1}{\sqrt{2}} \left( b_d + b_c \right) \\
a_2 &= \frac{1}{\sqrt{2}} \left( a_d - a_c \right) \\
b_2 &= \frac{1}{\sqrt{2}} \left( b_d - b_c \right)
\end{align*}\]
\[ b_i = S_{11} + S_{12} + S_{13} + S_{14} + S_{15} + a_2 \]

\[ \frac{1}{\sqrt{2}} \left( b_i + b_3 \right) = S_{16} \frac{1}{\sqrt{2}} \left( a_{d1} + a_{c1} \right) + S_{14} \frac{1}{\sqrt{2}} \left( a_{c1} - a_{d1} \right) + S_{15} \cdot a_2 \cdot \sqrt{2} \]  

(Eq. 1)

\[ b_i = S_{11} + S_{12} + S_{13} + a_2 + S_{15} \cdot a_2 \]

\[ \frac{1}{\sqrt{2}} \left( b_i - b_3 \right) = S_{21} \frac{1}{\sqrt{2}} \left( a_{d1} + a_{c1} \right) + S_{22} \frac{1}{\sqrt{2}} \left( a_{c1} - a_{d1} \right) + S_{23} \cdot a_2 \cdot \sqrt{2} \]  

(Eq. 2)

\[ (Eq. 1) + (Eq. 2) \Rightarrow 2bc_i = (S_{11} + S_{22}) \cdot (a_{d1} + a_{c1}) + (S_{12} + S_{21}) \cdot (a_{c1} - a_{d1}) + \sqrt{2} \left( S_{15} + S_{23} \right) \cdot a_2 \]

\[ \Rightarrow bc_i = \frac{1}{2} \left( S_{11} + S_{22} - S_{12} - S_{21} \right) \cdot a_{d1} + \frac{1}{2} \left( S_{12} + S_{21} + S_{11} + S_{22} \right) \cdot a_{c1} + \frac{\sqrt{2}}{2} \left( S_{15} + S_{23} \right) \cdot a_2 \]

\[ \Rightarrow \begin{cases} 
S_{bc11} = \frac{1}{2} \left( S_{11} + S_{22} - S_{12} - S_{21} \right) \\
S_{bc12} = \frac{1}{2} \left( S_{11} + S_{22} + S_{12} + S_{21} \right) \\
S_{bc13} = \frac{1}{2} \sqrt{2} \left( S_{15} + S_{23} \right)
\end{cases} \]

\[ (Eq. 1) - (Eq. 2) \Rightarrow 2bc_i = (S_{11} - S_{22}) \cdot (a_{d1} + a_{c1}) + (S_{12} - S_{21}) \cdot (a_{c1} - a_{d1}) + (S_{15} - S_{23}) \cdot \sqrt{2} \cdot a_2 \]

\[ \Rightarrow \begin{cases} 
b_i = \frac{1}{2} \left( S_{11} - S_{22} - S_{12} + S_{21} \right) \cdot a_{d1} + \frac{1}{2} \left( S_{11} - S_{22} + S_{12} - S_{21} \right) \cdot a_{c1} + \frac{\sqrt{2}}{2} \left( S_{15} - S_{23} \right) \cdot a_2 \\
b_i = S_{ad1} + S_{ac1} + S_{bc11} - S_{bc12} - S_{bc13}
\end{cases} \]

\[ \Rightarrow \begin{cases} 
S_{ad11} = \frac{1}{2} \left( S_{11} + S_{22} - S_{12} - S_{21} \right) \\
S_{ad12} = \frac{1}{2} \left( S_{11} + S_{22} - S_{12} - S_{21} \right) \\
S_{ad13} = \frac{1}{2} \sqrt{2} \left( S_{15} - S_{23} \right)
\end{cases} \]

. \[ b_3 = S_{31} + S_{32} + S_{33} + a_3 \Rightarrow b_3 = S_{31} \cdot \frac{1}{\sqrt{2}} \left( a_{d1} + a_{c1} \right) + S_{32} \cdot \frac{1}{\sqrt{2}} \left( a_{c1} - a_{d1} \right) + S_{33} \cdot a_2 \]

\[ \Rightarrow \begin{cases} 
b_3 = \frac{1}{\sqrt{2}} \left( S_{31} - S_{22} \right) \cdot a_{d1} + \frac{1}{\sqrt{2}} \left( S_{31} + S_{22} \right) \cdot a_{c1} + S_{33} \cdot a_2 \\
b_3 = S_{3d11} + S_{3d12} + S_{3d13} + a_{c1} + S_{3bc11} - S_{3bc12} - S_{3bc13}
\end{cases} \]

\[ \Rightarrow \begin{cases} 
S_{3d11} = \frac{1}{2} \left( S_{31} + S_{22} - S_{32} - S_{31} \right) \\
S_{3d12} = \frac{1}{2} \sqrt{2} \left( S_{31} - S_{22} \right) \\
S_{3d13} = \frac{1}{2} \sqrt{2} \left( S_{31} + S_{22} \right)
\end{cases} \]
\[
\begin{align*}
S_{SL21} &= \frac{1}{\sqrt{2}} (S_{S1} - S_{S2}) \\
S_{SC21} &= \frac{1}{\sqrt{2}} (S_{S1} + S_{S2}) \\
S_{SS22} &= S_{S3}
\end{align*}
\]

\[
\begin{align*}
S_{SD11} &= \frac{1}{\sqrt{2}} (S_{S1} + S_{S2} - S_{S1} - S_{S2}) \\
S_{SC11} &= \frac{1}{\sqrt{2}} (S_{S1} - S_{S2} - S_{S1} + S_{S2}) \\
S_{SD22} &= \frac{1}{\sqrt{2}} (S_{S1} + S_{S2} + S_{S1} + S_{S2}) \\
S_{SC21} &= \frac{1}{\sqrt{2}} (S_{S1} + S_{S2} - S_{S1} + S_{S2}) \\
S_{SS22} &= S_{S3}
\end{align*}
\]